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INVARIANT RADON TRANSFORMS ON A SYMMETRIC SPACE

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ABSTRACT. Injectivity and support theorems are proved for a class of Radon transforms, R_μ, for μ a smooth family of measures defined on a certain space of affine planes in $\mathbb{X}_0$, where $\mathbb{X}_0$ is the tangent space, of a Riemannian symmetric space of rank one. The transforms are defined by integrating against μ over these planes. We show that if $R_\mu f$ is supported inside a ball of radius R then so is f. This is true for $f \in L^2_c(\mathbb{X}_0)$ or $f \in \mathscr{E}'(\mathbb{X}_0)$. Furthermore, R_μ is invertible on either of these domains. The main technique is to use facts about spherical harmonics to reduce the problem to a one-dimensional integral equation.

0. INTRODUCTION

Radon transforms have been extensively studied. See, for example, Guillemin [2], Helgason [6], and Gelfand [1]. In this paper we will generalize and unify several results on Radon transforms. If a compact group K acts on a real vector space $\mathbb{X}_0$, then one has the following notion of a K-invariant Radon transform. The ingredients are Ξ_0, the space of affine hyperplanes, $F = \{(x, \xi) | x \in \xi\} \subset \mathbb{X}_0 \times \Xi_0$, and a smooth function μ on F. The Radon transform is defined for $f \in L^2_c(\mathbb{X}_0)$ by $R_\mu f(\xi) = \int_{x \in \xi} f(x)\mu(x, \xi)\, dx$. ($dx$ is Lebesgue measure.) If R_μ intertwines the actions of K it is called K invariant. This is easily seen to be equivalent to the condition $\mu(k \cdot x, k \cdot \xi) = \mu(x, \xi)$ for all $k \in K$ and $x \in \xi$.

It is natural to ask if R_μ is invertible and to look for support theorems. Quinto [11] covers the case when $K = O(n)$ and $\mathbb{X}_0 = \mathbb{R}^n$. He proves a support theorem and invertibility for K-invariant Radon transforms. Also, Helgason [3] proves support and injectivity theorems in the case $\mu \equiv 1$. Helgason [5] proves a support theorem for the classical Radon transform on Euclidean space.

If $\mathbb{X}_0$ has a natural complex or quaternionic structure one can play the same game with $\Xi_{\mathbb{F}}$, the space of complex or quaternionic hyperplanes. ($\mathbb{F}$ is either the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$.) Quinto [10] covers the case when $K = U(n)$ and $\mathbb{X}_0 = \mathbb{C}^n$. Once again he proves injectivity and a support theorem.

In this paper we will take $\mathbb{X}_0$ to be the tangent space of a rank one Rie-

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mannian symmetric space and K to be the isotropy subgroup. Our main results, Theorem 3.2 and Corollary 6.5, provide a support theorem and prove invertibility for K-invariant Radon transforms. We also obtain support and invertibility results for R_μ acting on distributions (Theorem 7.1 and Corollary 7.4).

We have used the group theory to present a unified picture of the transforms described above. Our technique of using the dual transform to reduce the proof of the support theorem to solving an Abel integral equation originates with Quinto [11]. Also, Helgason [3, pp. 105–112] uses an Abel integral equation to prove support theorems in the case $\mu \equiv 1$. One interesting feature of this is that the complex and quaternionic results follow from the real case. This is true because the spaces of real and complex affine hyperplanes are essentially isomorphic. (See Lemma 6.2.) The group theoretic approach also allows us to define Radon transforms on the symmetric space itself. There is also a natural generalization to higher rank spaces. Finally, one can investigate different representations of K. We will return to these questions in another paper.

The outline of the paper is as follows. In §1 we collect notation from the theory of symmetric spaces and we relate it to the familiar language of points and planes. Finally, we define the Radon transform and its dual. In §2 we justify the use of the term dual transform. In §3 we state our main theorem in the real case. §4 is devoted to reviewing known facts about harmonic polynomials, especially their decomposition under the action of K. In §5 we prove the main theorem. Using the results in §§2 and 4 the proof reduces to a one-dimensional integral equation of Volterra or Abel. In §6 we change fields. After relating the real and $\mathbb{F}$ hyperplane spaces and the real and $\mathbb{F}$ dual Radon transforms we extend the results of §3. Finally, in §7 we extend the results of §§3 and 6 to distributions.

1. Notation

We collect here the notation we will use throughout this paper. In the "flat case" we have the option of using either group theoretic notation or the more familiar terminology of points and planes. We list both notations, along with a brief dictionary relating the two. Also, note for the flat case the machinery of semisimple Lie groups is not really used (although it will play a role in later sections.)

We start with the standard notation for symmetric spaces. (See Helgason [4].) Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Fix K, a maximal compact subgroup of G and let $\mathfrak{k}$ denote its Lie algebra. We denote the Killing form on $\mathfrak{g}$ by $\langle\ ,\ \rangle$. Let $\mathfrak{p}$ be the orthogonal complement to $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$. Thus $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Under the adjoint action, K preserves $\mathfrak{p}$. We denote this action by a dot, that is, $\mathrm{Ad}(k)X = k \cdot X$ for $k \in K$ and $X \in \mathfrak{p}$. In general, we will use a dot to describe a group action whenever the intended action is clear from the context.

The Cartan motion group $K \ltimes \mathfrak{p}$ is denoted by G_0. Let $\mathbb{X} = G/K$ and

$\mathbb{X}_0 = G_0/K$. Then $\mathbb{X}$ and $\mathbb{X}_0$ are symmetric spaces. $\mathbb{X}$ is a Riemannian symmetric space of noncompact type and $\mathbb{X}_0 \approx \mathfrak{p}$ is its flat analogue. (We consider $\mathbb{X}_0$ to be the tangent space of $\mathbb{X}$ at the origin $o = eK \in G/K$.)

Fix $\mathfrak{a}$, a maximal abelian subalgebra of $\mathfrak{p}$. Let $M = Z_K(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in K, and let $M' = N_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{a}$ inside $\mathfrak{p}$ with respect to $\langle\ ,\ \rangle$. Let $P_\mathfrak{a}$ be the orthogonal projection to $\mathfrak{a}$.

The dual spaces of $\mathbb{X}$ and $\mathbb{X}_0$ are, respectively, Ξ, the space of horocycles in $\mathbb{X}$, and Ξ_0, the space of "flat horocycles" consisting of all G_0 translates of $\mathfrak{q}$. Thus, Ξ_0 is a set of affine planes in $\mathfrak{p}$. (If $\dim \mathfrak{a} = 1$ then Ξ_0 is the space of all affine hyperplanes.) As a homogeneous space $\Xi_0 = G_0/(M' \ltimes \mathfrak{q})$. To describe Ξ we need to choose an Iwasawa decomposition $G = KAN$ (see Helgason [4]). As a homogeneous space $\Xi = G/MN$.

In summary, we have

$$\begin{array}{llll} \mathfrak{g} = \mathfrak{k} + \mathfrak{p}, & G_0 = K \ltimes \mathfrak{p}, & \mathbb{X} = G/K, & \mathbb{X}_0 = G_0/K \approx \mathfrak{p}, \\ \mathfrak{p} = \mathfrak{a} + \mathfrak{q}, & M = Z_K(\mathfrak{a}), & \Xi = G/MN, & \Xi_0 = G_0/(M' \ltimes \mathfrak{q}), \\ P_\mathfrak{a} = \text{orthog. proj. to } \mathfrak{a}, & & M' = N_K(\mathfrak{a}). & \end{array}$$

For $\xi \in \Xi_0$ we let $\hat{\xi} = \{\mathbb{X} \in \mathfrak{p} | X \in \xi\}$. For $X \in \mathfrak{p}$ we let $\check{X} = \{\xi \in \Xi_0 | X \in \xi\}$. So, $\hat{\xi} =$ the points of ξ and $\check{X} =$ the set of flat horocycles through X. We make similar definitions for the curved spaces $\mathbb{X}$ and Ξ.

Consider the set $F_0 = \{(X, \xi) \in \mathbb{X}_0 \times \Xi_0 | X \in \xi\}$. It is easy to see that F_0 is isomorphic to G_0/M' via the map

$$gM' \to (g \cdot 0, g \cdot \mathfrak{q}). \tag{1.1}$$

Similarly, in the curved case we get $F = G/M$. Thus we have the double fibrations

$$\begin{array}{ccccccc} & G_0/M' = F_0 & & & & G/M = F & \\ {}^{p_1}\swarrow & & \searrow^{p_2} & & {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & \Xi & & X & & \Xi. \end{array}$$

Of course, $\check{X} = p_2 p_1^{-1}(X)$ and $\hat{\xi} = p_1 p_2^{-1}(\xi)$.

The set of flat horocycles through the origin is just

$$\check{0} = \{k \cdot \mathfrak{q} | k \in K\} \approx K/M'.$$

In general,

$$\check{X} = \{X + k \cdot \mathfrak{q} | k \in K\} \approx K/M'.$$

Thus, we have a canonical measure $d_X\xi$ on $\check{X}$ coming from the K invariant measure on K/M' (of total volume one). Likewise, the flat horocycle $\xi = Z + k \cdot \mathfrak{q}$ has

$$\hat{\xi} = \{Z + k \cdot Y | Y \in \mathfrak{q}\} \approx \mathfrak{q}.$$

Hence, it carries a canonical measure $d_\xi X$ coming from Lebesgue measure on $\mathfrak{q}$.

Let μ be a smooth function on F_0. Take $f \in C_c(\mathbb{X}_0)$ and $\varphi \in C(\Xi_0)$. We define the Radon transform and its dual by

$$R_\mu f(\xi) = \int_{X \in \hat{\xi}} f(X)\mu(X, \xi)\, d_\xi X,$$

$$R_\mu^* \varphi(X) = \int_{\xi \in \check{X}} \varphi(\xi)\mu(X, \xi)\, d_X \xi.$$

Let $\xi = Z + k \cdot \mathfrak{q}$. Using the descriptions of $\hat{\xi}$, $\check{X}$, and their measures given above we get:

$$R_\mu f(\xi) = \int_{Y \in \mathfrak{q}} f(Z + k \cdot Y)\mu(Z + k \cdot Y, Z + k \cdot \mathfrak{q})\, dY,$$

$$R_\mu^* \varphi(X) = \int_K \varphi(X + k \cdot \mathfrak{q})\mu(X, X + k \cdot \mathfrak{q})\, dk.$$

Here dk is the Haar measure on K of total volume one.

The Killing form provides a metric on $\mathfrak{p}$. For a plane in Ξ_0 its norm is defined as its distance from the origin. Thus $|k \cdot (H + \mathfrak{q})| = |H|$. We will let B^R denote the ball of radius R inside either $\mathfrak{p}$ or Ξ_0.

We now rewrite the Radon transforms using strictly group theoretic notation. For $Y \in \mathfrak{p}$ we let T_Y be the corresponding translation in G_0. Let $\xi_0 = e(M' \ltimes \mathfrak{q}) \in \Xi_0$. ($\xi_0$ is the horocycle $\mathfrak{q}$.) Fix a smooth function, μ, on $F_0 = G_0/M'$. For f and φ as before and $g \in G_0$ we have

$$R_\mu f(g \cdot \xi_0) = \int_{\mathfrak{q}} f(g \cdot T_Y \cdot 0)\mu(g \cdot T_Y M')\, dY,$$

and

$$R_\mu^* \varphi(g \cdot 0) = \int_K \varphi(g \cdot k \cdot \xi_0)\mu(g \cdot kM')\, dk.$$

This notation carries over easily to the curved case. Take $\mu \in C(G/M)$, $f \in C_c(\mathbb{X})$, and $\varphi \in C(\Xi)$. Then

$$R_\mu f(gMN) = \int_N f(gnK)\mu(gnM)\, dn,$$

$$R_\mu^* \varphi(gK) = \int_K \varphi(gkMN)\mu(gkM)\, dk.$$

From now on we will restrict our attention to the flat case. The curved case is more subtle. We hope to return to it in another paper.

Example. Let $G = \mathrm{SO}_e(n, 1)$. Then $K = \mathrm{SO}(n)$, $\mathbb{X}_0 = \mathfrak{p} \approx \mathbb{R}^n$, $\mathfrak{a}$ is any line through the origin, Ξ_0 is the space of all affine hyperplanes, and the action of K on $\mathfrak{p}$ is the usual one. For $\mu = 1$, R_μ is the classical Radon transform. If μ is K invariant (see §3) we are in the case of Quinto [11].

2. Duality

We have referred to R_μ and R_μ^* as dual transforms. In this section we will make this precise.

First we need "coordinates" and measures on $\mathbb{X}_0$ and Ξ_0. For $\mathbb{X}_0 = \mathfrak{p}$, we have Lebesgue measure dX. Clearly dX is G_0 invariant. We also have polar coordinates (see Helgason [4])

$$\Phi: K/M \times \mathfrak{a} \to \mathfrak{p} \quad \text{by } \Phi(kM, H) = k \cdot H.$$

The map Φ is surjective and generically w to one $(w = \#(M'/M))$.

For Ξ_0, we have polar coordinates

$$\Psi: K/M \times \mathfrak{a} \to \Xi_0 \quad \text{by } \Psi(kM, H) = k \cdot (H + \mathfrak{q}).$$

The map Ψ is surjective and generically w to one. Using these polar coordinates, we get a G_0-invariant measure on Ξ_0 given by

$$\int_{\Xi_0} \varphi(\xi)\, d\xi = \int_{K \times \mathfrak{a}} \varphi(k \cdot (H + \mathfrak{q}))\, dk\, dH.$$

The following lemma is proved in Helgason [7] in the case $\mu \equiv 1$.

Lemma 2.1. *The Radon transforms R_μ and R_μ^* are dual. That is, for $f \in L_c^2(\mathbb{X}_0)$ and $\varphi \in L^2(\Xi_0)$. (The subscript c indicates compact support.) We have*

$$\int_{\Xi_0} R_\mu f(\xi) \cdot \varphi(\xi)\, d\xi = \int_{\mathbb{X}_0} f(X) \cdot R_\mu^* \varphi(X)\, dX. \tag{2.1}$$

Proof. The left-hand side of equation (2.1) equals

$$\int_{K \times \mathfrak{a}} \int_{\mathfrak{q}} f(k(H+Y)) \cdot \varphi(k(H+\mathfrak{q}))\mu(k(H+Y), k(H+\mathfrak{q}))\, dY\, dk\, dH.$$

The right-hand side of (2.1) equals

$$\begin{aligned} &\int_{X_0} \int_K f(X) \cdot \varphi(X + k \cdot \mathfrak{q})\mu(X, X + k \cdot \mathfrak{q})\, dk\, dX \\ &= \int_K \int_{\mathbb{X}_0} f(k \cdot X) \cdot \varphi(k \cdot X + k \cdot \mathfrak{q})\mu(k \cdot X, k \cdot X + k \cdot \mathfrak{q})\, dX\, dk \\ &= \int_K \int_{\mathfrak{q}} \int_{\mathfrak{a}} f(k(H+Y)\varphi(k(H+Y+\mathfrak{q})) \\ &\qquad \times \mu(k(H+Y), k(H+Y+\mathfrak{q}))\, dY\, dH\, dk. \end{aligned}$$

The first equality follows from Fubini's theorem and the K invariance of the measure dX. Since $Y + \mathfrak{q} = \mathfrak{q}$ for $Y \in \mathfrak{q}$, the last formula equals the formula for the left-hand side given above. □

3. Main theorem

In this section we will state our main theorem. First we must define the notion of K invariance.

K invariance: Let L_k denote translation by $k \in K$ on $\mathbb{X}_0$ or on Ξ_0.

Lemma 3.1. *The following are equivalent for all* $k \in K$.

(1) $\mu(k \cdot X, k \cdot (X + \mathfrak{q})) = \mu(X, X + \mathfrak{q})$.
(2) *Considering* μ *as a function on* G_0/M', $\mu(k \cdot gM') = \mu(gM')$.
(3) $R_\mu(f \circ L_k) = (R_\mu f) \circ L_k$ *for* $f \in C_c(\mathfrak{p})$.
(4) $R^*_\mu(\varphi \circ L_k) = (R^*_\mu \varphi) \circ L_k$ *for* $\varphi \in C(\Xi_0)$.

Proof. (1) $\Leftrightarrow$ (2): This is obvious from equation (1.1).

(1) $\Leftrightarrow$ (3): We have

$$R_\mu(f \circ L_{k'})(X + k \cdot \mathfrak{q}) = \int_{\mathfrak{q}} f(k' \cdot (X + k \cdot Y))\mu(X + k \cdot Y, X + k \cdot \mathfrak{q})\, dY.$$

On the other hand,

$$(R_\mu f) \circ L_{k'}(X + k \cdot \mathfrak{q}) = \int_{\mathfrak{q}} f(k' \cdot (X + k \cdot Y))\mu(k' \cdot (X + k \cdot Y), k' \cdot (X + k \cdot \mathfrak{q}))\, dY.$$

Showing (1) $\Leftrightarrow$ (4) is similar. □

If any of the conditions in Lemma 3.1 hold we say that μ, R_μ, or R^*_μ is K invariant.

Assumptions. From now on we will assume G/K has real rank one. This means $\dim \mathfrak{a} = 1$. We also make the following assumptions:

(3.2)
(1) μ is K invariant.
(2) $\mu(H, H + \mathfrak{q}) \neq 0$ for all $H \in \mathfrak{a}$.
(3) μ is smooth.
(4) For all $H \in \mathfrak{a}$ and $Y \in \mathfrak{q}$, $\mu(H - Y, H + \mathfrak{q}) = \mu(H + Y, H + \mathfrak{q})$.

Remarks. (1) Combined with K invariance assumption (2) says $\mu(X, \xi) \neq 0$ if X is the point on ξ nearest the origin.

(2) Assumption (3) is not strictly necessary. By examining the proof of the main theorem we need only assume N times continuously differentiable for some fixed N.

(3) Assuming K invariance, we can state assumption (4) in a number of equivalent ways. For instance, $\mu(H + m'Y, H + \mathfrak{q}) = \mu(H + Y, H + \mathfrak{q})$ for all $H \in \mathfrak{a}$, $Y \in \mathfrak{q}$, and $m' \in M'$. Also, $\mu(T_H m' T_Y M') = \mu(T_H T_Y M')$. (See §1.) To show these two equivalences we need facts one and two stated between Lemma 4.1 and Lemma 4.2.

(4) Combined with K invariance assumption (4) is automatic if $m_{2\alpha} = 0$ or $m_{2\alpha} > 1$. (See §4.)

We can now state our main theorem. It is a generalization of Helgason's support theorem for the Euclidean Radon transform (see [5]). Our proof is modelled on Quinto [11].

Theorem 3.2 (Support Theorem). *Suppose* μ *satisfies the assumptions in* (3.2). *Then we have*

(1) *If* $f \in L^2_c(\mathbb{X}_0)$ *and* $\operatorname{supp} R_\mu f \subset B^R$ *then* $\operatorname{supp} f \subset B^R$.

(2) $R_\mu : L^2_c(\mathbb{X}_0) \to L^2(\Xi_0)$ *is injective.*

Remark. We assume a priori that f is compactly supported. However, this assumption is probably unnecessary. For example, for the classical Radon transform one need only assume a certain growth condition on f. (See Helgason [3].)

4. Structure theory

The proof of the support theorem requires some more facts and notation about symmetric spaces. In this section we will outline what is needed. For more detail see Helgason [4, 3].

The positive restricted roots of $\mathfrak{a}$ are denoted α and (if it exists) 2α. Their multiplicities are denoted m_α and $m_{2\alpha}$ and the corresponding root spaces are denoted $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$. We let θ be the Cartan involution corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. (That is, θ is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$.) We define

$$\mathfrak{q}_1 = \{X - \theta X | X \in \mathfrak{g}_\alpha\}, \quad \mathfrak{q}_2 = \{X - \theta X | X \in \mathfrak{g}_{2\alpha}\};$$

thus,

$$\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2 = \mathfrak{a}^\perp.$$

As before, $M = Z_K(\mathfrak{a})$ and $M' = N_K(\mathfrak{a})$. Also, we know that $W = M'/M$ has cardinality two.

We now fix $0 \neq H_0 \in \mathfrak{a}$. We know that the orbit $K \cdot H_0$ is isomorphic to K/M. Let $\mathscr{H}$ denote the harmonic polynomials on $\mathfrak{p}$. Let $\widehat{K}_M$ be the set of irreducible representations of K with a unique (up to scalar) M fixed vector. Then we have according to Kostant [8] (see also Helgason [3])

$$\mathscr{H} \hookrightarrow L^2(K/M)$$

and

$$\mathscr{H} \approx \bigoplus_{\delta \in \hat{K}_M} \mathscr{H}_\delta,$$

where the space $\mathscr{H}_\delta$ is an irreducible subrepresentation of $\mathscr{H}$ equivalent to δ and consisting of homogeneous polynomials of degree d_δ. The injection of $\mathscr{H}$ into $L^2(K/M)$ is accomplished by restricting a polynomial in $\mathscr{H}$ to the orbit $K \cdot H_0$. It commutes with the action of K and its image is dense in $L^2(K/M)$.

For each δ there is a unique M invariant polynomial $\varphi_\delta \in \mathscr{H}_\delta$ such that $\varphi(H_0) = 1$. We consider it as either a function of K/M or a harmonic polynomial on $\mathfrak{p}$.

The following "Funk-Hecke theorem" is trivial from the group theoretic viewpoint.

Lemma 4.1. *Choose any* $Y_\delta \in \mathscr{H}_\delta$ *(considered as a function on* K/M *by* $Y_\delta(k) = Y_\delta(k \cdot H_0)$*); then*

$$\int_M Y_\delta(k_1 m k_2)\, dm = Y_\delta(k_1)\varphi_\delta(k_2).$$

Proof. Fix $k_1 \in K$. Let $g(k_2) = \int_M Y_\delta(k_1 m k_2)\,dm$. Since $\mathscr{H}_\delta$ is K stable this function is in $\mathscr{H}_\delta$ and it is M invariant. Thus it is a multiple of φ_δ. To determine which multiple let $k_2 = e$. □

Finally, we need suitable coordinates on the sphere in $\mathfrak{p}$, $K \cdot H_0 = K/M$. We will use the $SU(2, 1)$ reduction introduced in Helgason [5]. Our calculations involving this reduction contain only minor modifications of calculations in [5]. Choose $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ and $0 \neq X_{2\alpha} \in \mathfrak{g}_{2\alpha}$. (If $m_{2\alpha} = 0$ then there is no $X_{2\alpha}$.) Define $Z_1 = X_\alpha - \theta X_\alpha$ and $Z_2 = X_{2\alpha} - \theta X_{2\alpha}$. We assume $|H_0| = |Z_1| = |Z_2| = 1$. We know (see Helgason [3]) that the subalgebra generated by H_0, Z_1, and Z_2 is isomorphic to $\mathfrak{su}(2, 1)$. (If $m_{2\alpha} = 0$ we get $\mathfrak{sl}(2)$.) From Kostant [8] we know the following:

(1) If $m_{2\alpha} > 1$ then M acts transitively on the product of the unit spheres in $\mathfrak{q}_1$ and $\mathfrak{q}_2$.

(2) If $m_{2\alpha} = 1$ then M acts transitively on the unit sphere in $\mathfrak{q}_1$.

(3) If $m_{2\alpha} \neq 0$ then $m_{2\alpha}$ is even and m_α is odd.

(4) If $m_{2\alpha} = 0$ then M is transitive on the sphere in $\mathfrak{q} = \mathfrak{q}_1$.

Thus we get the following coordinates on $K/M = K \cdot H_0$. (Recall that $|H_0| = 1$.)

$$(m, s, b) \to m(sH_0 + aZ_1 + bZ_2) \in K \cdot H_0.$$

Here, if $m_{2\alpha} > 1$ then

$$-1 \leq s \leq 1, \qquad 0 \leq b \leq \sqrt{1 - s^2}, \qquad m \in M, \qquad a = \sqrt{1 - s^2 - b^2};$$

if $m_{2\alpha} = 1$ then

$$-1 \leq s \leq 1, \qquad -\sqrt{1 - s^2} \leq b \leq \sqrt{1 - s^2}, \qquad m \in M, \qquad a = \sqrt{1 - s^2 - b^2};$$

if $m_{2\alpha} = 0$ then

$$-1 \leq s \leq 1, \qquad m \in M, \qquad a = \sqrt{1 - s^2}, \qquad b = 0.$$

Now we need to compute the measure on K/M in these coordinates.

Lemma 4.2. *For the various cases we get the following formulas for*

$$c \int_{K/M} f(kM)\,dk.$$

($c = A_{(m_\alpha + m_{2\alpha} + 1)}/A_{m_\alpha} A_{m_{2\alpha}}$, *where* A_n *is the area of the unit sphere in* $\mathbb{R}^n$.)

$m_{2\alpha} > 1$:

$$\int_{s=-1}^{1} \int_{b=0}^{\sqrt{1-s^2}} \int_M f(m(sH_0 + aZ_1 + bZ_2))a^{(m_\alpha - 2)} b^{(m_{2\alpha} - 1)}\,db\,ds\,dm,$$

$m_{2\alpha} = 1$:

$$\int_{s=s-1}^{1} \int_{b=-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \int_M f(m(sH_0 + aZ_1 + bZ_2)) a^{(m_\alpha - 2)} \, db \, ds \, dm,$$

$m_{2\alpha} = 0$:

$$\int_{s=-1}^{1} \int_M f(m(sH_0 + aZ_1)) a^{m_\alpha - 2} \, ds \, dm.$$

As before $a = \sqrt{1 - s^2 - b^2}$ ($\sqrt{1-s^2}$ *if* $m_{2\alpha} = 0$).

Remark. In the case $m_{2\alpha} = 0$ we have assumed $m_\alpha > 1$ The case $m_{2\alpha} = 0$ and $m_\alpha = 1$ is easy to handle separately.

Proof. This is a straightforward calculation using the fact that M is transitive on the product of the unit spheres of $\mathfrak{q}_1$ and $\mathfrak{q}_2$. □

We will need the following lemma.

Lemma 4.3. *If* $k \cdot H_0 = m(sH_0 + aZ_1 + bZ_2)$ *then for some* $m_1 \in M$ $k^{-1} \cdot H_0 = m_1(sH_0 - aZ_1 - bZ_2)$.

Proof. We reduce the proof to a calculation in $\mathfrak{su}(2, 1)$. Take $k_1 \in \mathrm{SU}(2, 1)$ (the copy of $\mathrm{SU}(2, 1)$ associated to (H_0, Z_1, Z_2)) such that $k_1 \cdot H_0 = sH_0 + aZ_1 + bZ_2$. This implies $k = mk_1m_2$ for some $m_2 \in M$. Thus, $k^{-1} = m_2^{-1}k_1^{-1}m^{-1}$ and $k^{-1} \cdot H_0 = m_2^{-1}k_1^{-1} \cdot H_0$. Now an easy calculation inside $\mathrm{SU}(2.1)$ shows $k_1^{-1} \cdot H_0 = m_3(sH_0 - aZ_1 - bZ_2)$ for some $m_3 \in M \cap \mathrm{SU}(2, 1)$. This proves the lemma. □

5. Proof of the support theorem

In this section we will prove the support theorem. First, we note that part (1) implies part (2) by letting $R \to 0$.

To prove part (1) we consider functions of the following form on Ξ_0:

$$g(k(H + \mathfrak{q})) = Y_\delta(k \cdot H_0) g_\delta(H) \tag{5.1}$$

where

(a) $Y_\delta \in \mathscr{H}_\delta$ and g_δ is smooth on $\mathfrak{a}$,
(b) $\operatorname{supp} g_\delta \cap B^R = \varnothing$,
(c) $g_\delta(-H) = (-1)^{d_\delta} g_\delta(H)$.

Condition (c) is needed to insure that g is well defined as a function on Ξ_0.

We will show that every function on $\mathfrak{p}$ of the form

$$h(k \cdot H) = Y_\delta(k \cdot H_0) h_\delta(H) \tag{5.2}$$

where

(a) $Y_\delta \in \mathscr{H}_\delta$,
(b) $\operatorname{supp} h_\delta \cap B^R = \varnothing$,

(c) $h_\delta(-H) = (-1)^{d_\delta} h_\delta(H)$,

(d) h_δ is smooth and compactly supported on $\mathfrak{a}$,

is $R_\mu^* g$ for some g of the type (5.1) above.

Assuming this, part (1) follows directly: Pick any h of type (5.2) and take g of type (5.1) such that $R_\mu^* g = h$. Then, since $\operatorname{supp} R_\mu f \subset B^R$ we have from Lemma 2.1

$$0 = \langle R_\mu f, g\rangle = \langle f, R_\mu^* g\rangle = \langle f, h\rangle.$$

But the span of the functions h of type (5.2) (δ is allowed to vary) is dense in $L^2(\mathbb{X}_0 - B^R)$. Thus f is perpendicular to $L^2(\mathbb{X}_0 - B^R)$. This implies $\operatorname{supp} f \subset B^R$, proving part (1).

We now have to show that $R_|^* \mu$ maps functions of type (5.1) onto those of type (5.2). (Because of the condition of compact support in (5.2) the image is larger than just functions of type (5.2).) We will do this by explicitly reducing the integral equation $R_\mu^* g = h$ to a one-dimensional equation of Volterra type.

Pick h of type (5.2). We need to solve $R_\mu^* g = h$ for g of type (5.1). That is, we must solve

$$h(k_0 \cdot H) = Y_\delta(k_0 \cdot H_0) h_\delta(H) = R_\mu^* g(k_0 \cdot H). \tag{5.3}$$

Computing, we get

$$\begin{aligned}
R_\mu^* g(k_0 \cdot H) &= \int_{K/M} g(k_0 \cdot H + k \cdot \mathfrak{q}) \mu(k_0 \cdot H, k_0 \cdot H + k \cdot \mathfrak{q})\, dk \\
&= \int_{K/M} g(k_0 \cdot H + k_0 k \cdot \mathfrak{q}) \mu(k_0 \cdot H, k_0 \cdot H + k_0 k \cdot \mathfrak{q})\, dk \\
&= \int_{K/M} g(k_0 k(k^{-1} \cdot H + \mathfrak{q}) \mu(k_0 \cdot H, k_0 k(k^{-1} \cdot H + \mathfrak{q}))\, dk \\
&= \int_{K/M} g(k_0 k(P_\mathfrak{a}(k^{-1} \cdot H) + \mathfrak{q}) \mu(H, k(P_\mathfrak{a}(k^{-1} \cdot H) + \mathfrak{q}))\, dk \\
&= \int_{K/M} Y_\delta(k_0 k \cdot H_0) g_\delta(P_\mathfrak{a}(k^{-1} \cdot H)) \mu(k^{-1} \cdot H, P_\mathfrak{a}(k^{-1} \cdot H) + \mathfrak{q})\, dk \\
&= \int_{K/M} \int_M Y_\delta(k_0 mk \cdot H_0) g_\delta(P_\mathfrak{a}(k^{-1} m^{-1} \cdot H)) \\
&\qquad\qquad \times \mu(k^{-1} m^{-1} \cdot H, P_\mathfrak{a}(k^{-1} m^{-1} \cdot H) + \mathfrak{q})\, dm\, dk \\
&= \int_{K/M} \int_M Y_\delta(k_0 mk \cdot H_0) g_\delta(P_\mathfrak{a}(k^{-1} \cdot H)) \\
&\qquad\qquad \times \mu(k^{-1} \cdot H, P_\mathfrak{a}(k^{-1} \cdot H) + \mathfrak{q})\, dm\, dk \\
&= \int_{K/M} Y_\delta(k_0 \cdot H_0) \varphi_\delta(k \cdot H_0) g_\delta(P_\mathfrak{a}(k^{-1} \cdot H)) \\
&\qquad\qquad \times \mu(k^{-1} \cdot H, P_\mathfrak{a}(k^{-1} \cdot H) + \mathfrak{q})\, dk.
\end{aligned}$$

The second and sixth equalities follow from the K invariance of dk. The fourth and fifth equalities follow from the K invariance of μ. The seventh follows because $m \cdot H = H$. The last equality is Lemma 4.1.

Let $H = rH_0$. The last integral above then equals

$$Y_\delta(k_0 \cdot H_0) \int_{K/M} \varphi_\delta(k \cdot H_0) g_\delta(rP_{\mathfrak{a}}(k^{-1} \cdot H_0)) \mu(rk^{-1} \cdot H_0, rP_{\mathfrak{a}}(k^{-1} \cdot H_0) + \mathfrak{q})\, dk.$$

Thus, equation (5.3) reduces to

$$h_\delta(rH_0) = \int_{K/M} \varphi_\delta(k \cdot H_0) g_\delta(rP_{\mathfrak{a}}(k^{-1} \cdot H_0)) \mu(rk^{-1} \cdot H_0, rP_{\mathfrak{a}}(k^{-1} \cdot H_0) + \mathfrak{q})\, dk.$$

Using Lemmas 4.3 and 4.2 this equation becomes the following:

$m_{2\alpha} > 1$:

$$\begin{aligned} h_\delta(rH_0) = c \int_M \int_{s=-1}^{1} \int_{b=0}^{\sqrt{1-s^2}} & \varphi_\delta(m(sH_0 + aZ_1 + bz_2)) g_\delta(rsH_0) \\ & \times \mu(m_1 r(sH_0 - aZ_1 - bZ_1), rsH_0 + \mathfrak{q}) a^{m_\alpha - 2} b^{m_{2\alpha} - 1}\, db\, ds\, dm, \end{aligned}$$

$m_{2\alpha} = 1$:

$$\begin{aligned} h_\delta(rH_0) = c \int_M \int_{s=-1}^{1} \int_{b=-\sqrt{1-s^2}}^{\sqrt{1-s^2}} & \varphi_\delta(m(sH_0 + aZ_1 + bZ_2)) \\ & \times g_\delta(rsH_0) \mu(m_1 r(sH_0 - aZ_1 - bZ_2), rsH_0 + \mathfrak{q}) a^{m_\alpha - 2}\, db\, ds\, dm, \end{aligned}$$

$m_{2\alpha} = 0$:

$$\begin{aligned} h_\delta(rH_0) = c \int_M \int_{s=-1}^{1} & \varphi_\delta(m(sH_0 + az_1)) \\ & \times g_\delta(rsH_0) \mu(m_1 r(sH_0 - aZ_1), rsH_0 + \mathfrak{q}) a^{m_\alpha - 2}\, ds\, dm. \end{aligned}$$

Because of the homogeneity requirements on h_δ and g_δ we can assume that $r > 0$.

We now manipulate these equations as follows:

$m_{2\alpha} > 1$:

$$\begin{aligned} h_\delta(rH_0) &= c \int_{s=-1}^{1} \int_{b=0}^{\sqrt{1-s^2}} \varphi_\delta((sH_0 + aZ_1 + bZ_2)) g_\delta(rsH_0) \\ &\qquad \times \mu(r(sH_0 - aZ_1 - bZ_2), rsH_0 + \mathfrak{q}) a^{m_\alpha - 2} b^{m_{2\alpha} - 1}\, db\, ds \\ &= 2c \int_{s=0}^{1} \int_{b=0}^{\sqrt{1-s^2}} \varphi_\delta((sH_0 + aZ_1 + bZ_2)) g_\delta(rsH_0) \\ &\qquad \times \mu(r(sH_0 - aZ_1 - bZ_2), rsH_0 + \mathfrak{q}) a^{m_\alpha - 2} b^{m_{2\alpha} - 1}\, db\, ds. \end{aligned}$$

The first equality follows from the M invariance of φ_δ and μ and the fact that M centralizes $\mathfrak{a}$. The second equality follows from the homogeneity of φ_δ and g_δ, the M invariance of φ_δ, the transitivity of M on the product of the unit spheres of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, and the M' invariance of μ.

Similarly, for $m_{2\alpha} = 1$,

$$h_\delta(rH_0) = 2c\int_{s=0}^{1}\int_{b=0}^{\sqrt{1-s^2}} [\varphi_\delta((sH_0 + aZ_1 + bZ_2)) + \varphi_\delta((sH_0 + aZ_1 - bZ_2))] \times g_\delta(rsH_0)\mu(r(sH_0 - aZ_1 - bZ_2), rsH_0 + \mathfrak{q})a^{m_\alpha - 2}\, db\, ds.$$

Here we used all the invariance used above and the hypothesis $\mu(H+Y, H+\mathfrak{q}) = \mu(H - Y, H + \mathfrak{q})$.

Finally, for $m_{2\alpha} = 0$ (assuming $m_\alpha > 1$) we get

$$h_\delta(rH_0) = 2c\int_{s=0}^{1} \varphi_\delta((sH_0 + aZ_1))g_\delta(rsH_0)\mu(r(sH_0 - aZ_1), rsH_0 + \mathfrak{q})a^{m_\alpha - 2}\, ds.$$

Now make the change of coordinates $y = rs$, $\beta = rb$ and let $\alpha^2 = r^2a^2 = r^2 - y^2 - \beta^2$. Using the homogeneity of φ and the assumption that g_δ is supported outside the ball of radius R we get:
For $m_{2\alpha} > 1$:

$$h_\delta(rH_0) = 2c\int_{y=R}^{r}\int_{\beta=0}^{\sqrt{r^2-y^2}} \varphi_\delta(yH_0 + \alpha Z_1 + \beta Z_2)g_\delta(yH_0) \times \mu(yH_0 - \alpha Z_1 - \beta Z_2, yH_0 + \mathfrak{q})\alpha^{m_\alpha - 2}\beta^{m_{2\alpha}-1}r^{-(m_\alpha + m_{2\alpha} - 1 + d_\delta)}\, dy\, d\beta.$$

For $m_{2\alpha} = 1$:

$$h_\delta(rH_0) = 2c\int_{y=R}^{r}\int_{\beta=0}^{\sqrt{r^2-y^2}} [\varphi_\delta(yH_0 + \alpha Z_1 + \beta Z_2) + \varphi_\delta(yH_0 + \alpha Z_1 - \beta Z_2)] \times g_\delta(yH_0)\mu(yH_0 - \alpha Z_1 - \beta Z_2, yh_0 + \mathfrak{q})\alpha^{m_\alpha - 2}r^{-(m_\alpha + d_\delta)}\, dy\, d\beta.$$

For $m_{2\alpha} = 0$:

$$h_\delta(rH_0) = 2c\int_{y=R}^{r} \varphi_\delta(yH_0 + \alpha Z_1)g_\delta(yH_0) \times \mu(yH_0 - \alpha Z_1, yH_0 + \mathfrak{q})\alpha^{m_\alpha - 2}r^{-(m_\alpha - 1 + d_\delta)}\, dy.$$

Now let $u^2 = r^2 - y^2$ (in particular, for $m_{2\alpha} = 0$ we get $u^2 = \alpha^2$). Use the equations to define the function $\Phi(u, y)$ such that the following equation holds (for any value of $m_{2\alpha}$).

$$r^{(m_\alpha + m_{2\alpha} - 1 + d_\delta)}h_\delta(rH_0) = \int_R^r \Phi(\sqrt{r^2 - y^2}, y)g_\delta(yH_0)\, dy.$$

The following Lemma 5.1 and Theorem 5.2 show that if $h_\delta(rH_0)$ is smooth in r and compactly supported in $R \le r \le \infty$ then we can solve the above equation for g_δ. This completes the proof of the support theorem.

Lemma 5.1. *We can write* $\Phi(u, y) = u^{m_\alpha + m_{2\alpha} - 2} E(u^2, y)$, *where* E *is smooth and* $E(0, r) \neq 0$.

Proof. Case a, $m_{2\alpha} \neq 0$. By definition

$$\Phi(u, y) = 2c \int_0^u F(y, \sqrt{u^2 - \beta^2}, \beta)\mu_1(y, -\sqrt{u^2 - \beta^2}, -\beta) \times (u^2 - \beta^2)^{(m_\alpha - 2)/2} \beta^{m_{2\alpha} - 1} \, d\beta,$$

where $F(y, \sqrt{u^2 - \beta^2}, \beta)$ equals

$$\varphi_\delta(yH_0 + (\sqrt{u^2 - \beta^2})Z_1 + \beta Z_2) \quad \text{if } m_{2\alpha} > 1, \quad \text{and}$$
$$\varphi_\delta(yH_0 + (\sqrt{u^2 - \beta^2})Z_1 + \beta Z_2) + \varphi_\delta(yH_0 + (\sqrt{u^2 - \beta^2})Z_1 - \beta Z_1) \quad \text{if } m_{2\alpha} = 1.$$

Also $\mu_1(y, -\sqrt{u^2 - \beta^2}, -\beta) = \mu(yH_0 - (\sqrt{u^2 - \beta^2})Z_1 - \beta Z_2, yH_0 + \mathfrak{q})$.

Using the M invariance of φ_δ and the facts about the transitivity of M stated between Lemmas 4.1 and 4.2 we see that F is even in its second and third variables. Using the M invariance of μ (and if $m_{2\alpha} = 1$, hypothesis (3.2)(4)) we see that μ_1 is also even in its second and third variables. Thus, making the obvious change in notation we get

(5.4)
$$\Phi(u, y) = \int_0^u \tilde{F}(y, u^2 - \beta^2, \beta^2)\tilde{\mu}(y, u^2 - \beta^2, \beta^2)(u^2 - \beta^2)^{(m_\alpha - 2)/2} \beta^{m_{2\alpha} - 1} \, d\beta,$$

Since φ_δ is a polynomial, $\tilde{F}$ is smooth. By hypothesis $\tilde{\mu}$ is smooth. Thus $\Phi(u, y)$ is also smooth. Also note, since $m_{2\alpha} \neq 0$, that m_α is even.

Using equation (5.4) we differentiate $\Phi(u, y)$, with respect to u, $(m_\alpha + m_{2\alpha} - 2)$ times. We get

(5.5)
$$\left(\frac{\partial}{\partial u}\right)^{(m_\alpha + m_{2\alpha} - 2)} \Phi(0, y) = c\tilde{F}(y, 0, 0)\cdot\tilde{\mu}(y, 0, 0) = c\varphi_\delta(yH_0)\cdot\mu(yH_0, yH_0 + \mathfrak{q})$$

for some nonzero constant c. (To compute c: note that

$$c = \left(\frac{\partial}{\partial u}\right)^{(m_\alpha + m_{2\alpha} - 2)} \int_0^u (u^2 - \beta^2)^{(m_\alpha - 2)/2} \beta^{m_{2\alpha} - 1} \, d\beta.$$

The integral is easily computed to be $c_1 u^{(m_\alpha + m_{2\alpha k} - 2)}$ (m_α is even). The constant c_1 is not zero since the integrand is positive. Thus $c = c_1(m_\alpha + m_{2\alpha} - 2)! \neq 0$.) Because of the assumption (3.2)(2) equation (5.5) is never equal to zero. On the other hand all lower-order derivatives with respect to u at $u = 0$ are easily seen to be 0. This proves Case a.

Case b, $m_{2\alpha} = 0$, is immediate from the assumption (3.2). □

Theorem 5.2. *Let* $K(r, y)$ *be a smooth function such that* $K(r, r) \neq 0$ *for all* $r \geq 0$. *Let* k *be a nonnegative integer and let* ψ *be a smooth function with*

$\operatorname{supp}\psi \subseteq [R, \infty]$. *Then the equation* $\psi(r) = \int_R^r (r-y)^{k/2} K(r, y) g(y)\, dy$ *can always be solved for a unique* $g(y)$, *which is smooth and with* $\operatorname{supp} g \subseteq [R, \infty]$.

Proof. If k is even this is a Volterra equation of the first kind. If k is odd this is Abel's equation. See Yosida [12]. □

6. Change of fields

So far we have used real planes in $\mathfrak{p}$. In this section we will investigate changing fields. That is, if $\mathfrak{p}$ has a complex or quaternionic structure then we can take the space of complex or quaternionic hyperplanes to be our horocycle space.

Remark. In $\mathbb{H}^n$ we will have scalars act on the right. That is, $a \cdot v = v\bar{a}$, for $a \in \mathbb{H}$ and $v \in \mathbb{H}^n$.

We consider two series of symmetric spaces, G/K:

(1) $G = \mathrm{SU}(n, 1)$, $K = S(U(n) \times U(1))$.

(2) $G = \mathrm{Sp}(n, 1)$, $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

We write $\mathbb{F} = \mathbf{C}$ in case (1) and $\mathbb{F} = \mathbb{H}$ in case (2).

Remark. There are other possibilities. For example, we have a complex structure on $\mathbb{H}^n$. Also, we have an octonionic structure on $F_4/\mathrm{SO}(9)$.

We can make the following identifications:

$$\mathbb{X}_0 = \mathfrak{p} = \mathbb{F}^n, \qquad H_0 = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} \in \mathfrak{p}, \qquad \mathfrak{a} = \mathbb{R} \cdot H_0,$$

$$\mathfrak{a} + \mathfrak{q} = \mathbb{F} \cdot H_0, \qquad \mathfrak{q}_1 = \left\{ \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_{n-1} \\ 0 \end{pmatrix} \middle| f_i \in \mathbb{F} \right\} \quad \text{(an } \mathbb{F} \text{ hyperplane)}.$$

We consider $\mathrm{Sp}(n)$ to be the group of $n \times n$ matrices with entries in $\mathbb{H}$ preserving the inner product on $\mathbb{H}^n$ given by

$$\langle v, w \rangle = \sum \bar{w}_i v_i$$

Take $k = (k_1, u) \in K$ (recall, K is a subgroup of a product group) and $X \in \mathfrak{p}$; then, using the identifications above, we have $k \cdot X = k_1 X \bar{u}$. ($k_1 X$ is matrix multiplication)

We let $\Xi_{\mathbb{F}}$ denote the space of affine $\mathbb{F}$ hyperplanes. It is clear that G_0 acts transitively on $\Xi_{\mathbb{F}}$. This makes $\Xi_{\mathbb{F}}$ a homogeneous space. Define the map

$$\Phi\colon \Xi_0 \to \Xi_{\mathbb{F}}$$

by

$$\Phi(k \cdot (H + \mathfrak{q})) = k \cdot (H + \mathfrak{q}_1).$$

Lemma 6.1. *The map* Φ *is well defined.*

Proof. If $k \cdot (H + \mathfrak{q}) = k_1 \cdot (H_1 + \mathfrak{q})$ then $km = k_1$ and $m \cdot H_1 = H$ for some $m \in M'$. Since $m \cdot \mathfrak{q}_1 = \mathfrak{q}_1$ we get $k \cdot (H + \mathfrak{q}_1) = k_1 \cdot (H_1 + \mathfrak{q}_1)$. □

Lemma 6.2. *The map*

$$\Phi \colon \Xi_0/\{\textit{hyperplanes through } 0\} \to \Xi_{\mathbb{F}}/\{\mathbb{F} \textit{ hyperplanes through } 0\}$$

is a diffeomorphism.

Proof. Suppose $H \neq 0$ and $H_1 \neq 0$. (So, $0 \notin k \cdot H + \mathfrak{q})$ etc.) If $k \cdot (H + \mathfrak{q}_1) = k_1 \cdot (H_1 + \mathfrak{q}_1)$ then $k_1^{-1} k \cdot (H + \mathfrak{q}_1) = H_1 + \mathfrak{q}_1$. Thus, $k_1^{-1} k \cdot \mathfrak{q}_1 = \mathfrak{q}_1$ and $k_1^{-1} k \cdot H = H_1 + Y$ for some $Y \in \mathfrak{q}_1$. This implies $|H_1| \leq |H|$. By symmetry, $|H| = |H_1|$. This implies $k_1^{-1} k \cdot H = H_1$. Since $H \neq 0$ we have $k_1^{-1} k \in M'$. Thus, $k_1^{-1} k \cdot \mathfrak{q} = \mathfrak{q}$ and, hence, $k \cdot (H + \mathfrak{q}) = k_1 \cdot (H_1 + \mathfrak{q})$. This shows Φ is injective.

Let $\widetilde{M}$ be the stabilizer of $\mathfrak{q}_1$ in K. To show Φ is surjective we simply have to note that $\widetilde{M}$ acts transitively on the unit sphere in $\mathfrak{a} + \mathfrak{q}_2$.

To show Φ is regular we use the following coordinates. For $0 \neq H \in \mathfrak{a}$ take a small neighborhood $U \subset \mathfrak{a}$. Then $K/M \times U$ $((k, H) \to k \cdot (H + \mathfrak{q})$ etc.) provides local coordinates for both spaces. With respect to these coordinates Φ is the identity map. Thus, Φ is regular. □

This lemma shows that for purposes of integration we can use the "coordinates" $K/M \times \mathfrak{a}$ on $\Xi_{\mathbb{F}}$. The measure we will use is $c|H|^{m_{2\alpha}}\, dk\, dH$. (It is G_0 invariant.) Here, c is the area of the unit sphere in $\mathfrak{a} + \mathfrak{q}_2$.

The next proposition is best developed using group theoretic notation. As before, let $\widetilde{M}$ be the stabilizer of $\mathfrak{q}_1$ in K. Similar to the real case, we have

$$G_0/\widetilde{M} \approx \{(X, \xi) | X \in \xi\} \qquad (g\widetilde{M} \leftrightarrow (g \cdot 0, g \cdot \mathfrak{q}_1)).$$

This gives the double fibration

$$\begin{array}{ccc} & G_0/\widetilde{M} & \\ {}^{p_1}\swarrow & & \searrow{}^{p_2} \\ \mathbb{X}_0 & & \Xi_{\mathbb{F}}. \end{array}$$

As in §1, we get $\check{X} = p_2 p_1^{-1}(X)$ and $\hat{\xi} = p_1 p_2^{-1}(\xi)$.

For $\mu \in C(G_0/\widetilde{M})$ we define

$${}_{\mathbb{F}}R_\mu f(\xi) = \int_{\mathfrak{q}_1} f(g \cdot T_Y \cdot 0)\mu(g \cdot T_Y \widetilde{M})\, dY$$

and

$${}_{\mathbb{F}}R_\mu^* \varphi(X) = \int_K \varphi(T_X k \cdot \mathfrak{q}_1)\mu(T_X k \widetilde{M})\, dk.$$

The next lemma describes the duality between ${}_{\mathbb{F}}R_\mu$ and ${}_{\mathbb{F}}R_\mu^*$.

Lemma 6.3. *Let dX be the usual measure on $\mathfrak{p}$ and, as before, let*

$$c|H|^{m_{2\alpha}}\, dH\, dk = d\xi$$

be the measure on $\Xi_{\mathbb{F}}$. Then

$$\int_{\Xi_{\mathbb{F}}} {}_{\mathbb{F}}R_\mu f(\xi)\varphi(\xi)\, d\xi = \int_{\mathfrak{p}} f(X)\, {}_{\mathbb{F}}R^*_\mu \varphi(X)\, dX.$$

Proof. The right-hand side is

$$\begin{aligned}
&\int_{\mathfrak{p}}\int_K f(X)\varphi(X + k\cdot \mathfrak{q}_1)\mu(T_X k\widetilde{M})\, dk\, dX \\
&\quad = \int_K\int_{\mathfrak{p}} f(k\cdot X)\varphi(k\cdot (X+\mathfrak{q}_1)\mu(T_{k\cdot X}k\widetilde{M})\, dX\, dk \\
&\quad = \int_K\int_{\mathfrak{a}+\mathfrak{q}_2}\int_{\mathfrak{q}_1} f(k\cdot(H+Y_2+Y_1))\varphi(k\cdot(H+Y_2+\mathfrak{q}_1)) \\
&\qquad\qquad\qquad \times \mu(T_{H+Y_2+Y_1}\widetilde{M})\, dH\, dY_2\, dY_1\, dk \\
&\quad = c\int_K\int_{\mathfrak{a}}\int_{\widetilde{M}}\int_{\mathfrak{q}_1} f(k\cdot(\tilde{m}\cdot H+Y_1))\varphi(k\cdot(\tilde{m}\cdot H+\mathfrak{q}_1)) \\
&\qquad\qquad\qquad \times \mu(T_{\tilde{m}\cdot H+Y_1}\widetilde{M})|H|^{m_{2\alpha}}\, d\tilde{m}\, dH\, dY_1\, dk \\
&\quad = c\int_K\int_{\mathfrak{a}}\int_{\widetilde{M}}\int_{\mathfrak{q}_1} f(k\cdot\tilde{m}\cdot(H+Y_1))\varphi(k\cdot\widetilde{M}\cdot(H+\mathfrak{q}_1)) \\
&\qquad\qquad\qquad \times \mu(T_{H+Y_1}\widetilde{M})|H|^{m_{2\alpha}}\, d\tilde{m}\, dH\, dY_1\, dk \\
&\quad = c\int_K\int_{\mathfrak{a}}\int_{\mathfrak{q}_1} f(k\cdot(H+Y_1))\varphi(k\cdot(H+\mathfrak{q}_1))\mu(T_{H+Y_1}\widetilde{M})|H|^{m_{2\alpha}}\, dH\, dY_1\, dk \\
&\quad = \int_{\Xi_{\mathbb{F}}} {}_{\mathbb{F}}R_\mu f(\xi)\varphi(\xi)\, d\xi. \quad \square
\end{aligned}$$

Since $\widetilde{M} \supset M'$ we have

$$\mu \in C(G_0/\widetilde{M}) \subset C(G_0/M').$$

(We consider μ as a function on G_0 invariant by $\widetilde{M}$.) Thus, we also have R^*_μ acting on functions on Ξ_0. The next lemma makes the connection between transforms. From now on we will consider μ as a function on G_0 with certain invariance properties.

Lemma 6.4. *For any $\varphi \in C(\Xi_0)$ there is a $\tilde{\varphi} \in C(\Xi_{\mathbb{F}})$ such that $R^*_\mu\varphi = {}_{\mathbb{F}}R^*_\mu\tilde{\varphi}$. If φ is smooth then so is $\tilde{\varphi}$.*

Proof. Let $\tilde{\varphi}(g\cdot\mathfrak{q}_1) = \int_{\widetilde{M}} \varphi(g\tilde{m}\cdot\mathfrak{q})\, d\tilde{m}$. Clearly $\tilde{\varphi}$ is defined on $G_0/(\widetilde{M} \ltimes \mathfrak{q}_1) =$

$\Xi_{\mathbb{F}}$. Also,

$$\begin{aligned}{}_{\mathbb{F}}R_\mu^*\tilde{\varphi}(X) &= \int_K \tilde{\varphi}(T_X k \cdot \mathfrak{q}_1)\mu(T_X k)\, dk \\ &= \int_K \int_{\widetilde{M}} \varphi(T_X k\tilde{m} \cdot \mathfrak{q})\mu(T_X k)\, d\tilde{m}\, dk \\ &= \int_{\widetilde{M}} \int_K \varphi(T_X k \cdot \mathfrak{q})\mu(T_X k\tilde{m}^{-1})\, d\tilde{m}\, dk \\ &= \int_K \varphi(T_X k \cdot \mathfrak{q})\mu(T_X k)\, dk \\ &= R_\mu^*\varphi(X).\end{aligned}$$

The third equality follows from the $\widetilde{M}$ invariance dk and the fourth is from the $\widetilde{M}$ invariance of μ.

The second statement in the lemma is clear. □

Remark. Clearly, if $\operatorname{supp}\varphi \cap B^R = \varnothing$ then $\operatorname{supp}\tilde{\varphi} \cap B^R = \varnothing$. Here, the balls are in Ξ_0 and $\Xi_{\mathbb{F}}$ respectively. (Recall, the norm of a plane in either space is defined to be the distance of the plane to the origin.)

The following corollary is the main result in this section, it generalizes Quinto [10].

Corollary 6.5. *Suppose* μ *satisfies*

(1) μ *is smooth.*
(2) μ *is* K *invariant* $(\mu \in C^\infty(K\backslash G_0/\widetilde{M}))$.
(3) $\mu(T_H) \neq 0$ *for all* $H \in \mathfrak{a}$.

Then (1) *if* $f \in L_c^2(\mathbb{X}_0)$ *and* $\operatorname{supp}\ {}_{\mathbb{F}}R_\mu f \subset B^R$ *then* $\operatorname{supp} f \subset B^R$,

(2) ${}_{\mathbb{F}}R_\mu$ *is injective on* $L_c^2(\mathbb{X}_0)$.

Proof. Part (1): The proof proceeds just like that of Theorem 3.2. By Lemma 6.4 and the proof of Theorem 3.2 every function of type (5.2) $(h(k \cdot H) = Y_\delta(k \cdot H_0)h_\delta(H)\cdots)$ is ${}_{\mathbb{F}}R_\mu^*\tilde{\varphi}$ for some φ of type (5.1).

Suppose $\operatorname{supp}{}_{\mathbb{F}}R_\mu\tilde{\varphi} \subset B^R$, h is of type (5.2), and φ is chosen as above. Then,

$$\langle f, h\rangle = \langle f, {}_{\mathbb{F}}R_\mu^*\tilde{\varphi}\rangle = \langle {}_{\mathbb{F}}R_\mu f, \tilde{\varphi}\rangle = 0.$$

The third bracket is the L^2 inner product with respect to the measure on $\Xi_{\mathbb{F}}$ from Lemma 6.3. It equals zero by the remark following Lemma 6.4.

As in Theorem 3.2, part (2) follows from part (1) by letting R go to zero. □

Remarks. (1) As before, the condition $\mu(T_H) \neq 0$ combined with K invariance says that $\mu(X, \xi) \neq 0$ if X is the point of ξ nearest the origin.

(2) It is interesting to note that Quinto [10] reduces the proof of the support theorem to a Volterra equation and we reduce it to Abel's equation. The difference comes from the way the $\widetilde{M}$ invariance is applied.

7. The support theorem for distributions

In this section we will extend the support theorem to distributions. Most likely, Quinto's argument via elliptic operators (Quinto [11]) would carry through here. However, we will use a different approach via the representation theory of K on $\mathfrak{p} = \mathbb{X}_0$.

For $T \in \mathscr{E}'(\mathfrak{p})$ we define $R_\mu T(\varphi) = T(R_\mu^* \varphi)$. This is well defined since R_μ^* is continuous on $C^\infty(\Xi_0)$.

Part (1) of the following theorem appears in [3, Theorem 2.20] in the case $\mu \equiv 1$. The entire theorem is a generalization of a result in [10].

Theorem 7.1. *Suppose* μ *satisfies the hypothesis in Theorem* 3.2 (*see equation* (3.2)). *Then*

(1) *If* $T \in \mathscr{E}'(\mathbb{X}_0)$ *and* $\operatorname{supp} R_\mu T \subset B^R$ *then* $\operatorname{supp} T \subset B^R$.
(2) *The operator* $R_\mu \colon \mathscr{E}'(\mathfrak{p}) \to \mathscr{E}'(\Xi_0)$ *is injective.*

Proof. Part (1): The proof of Theorem 3.2(1) shows R_μ^* maps functions in $C^\infty(\Xi_0)$ supported outside the ball of radius R onto a dense subspace of functions in $C^\infty(\mathbb{X}_0)$ supported outside the ball of radius R. This proves part (1).

For part (2), by letting $R \to 0$, it is enough to show that

$$R_\mu \colon \text{Distributions supported at } 0 \to \mathscr{E}'(\Xi_0)$$

is injective. Let $Tf = Df(0)$ be an arbitrary distribution supported at the origin. (D is a constant coefficient differential operator on $\mathfrak{p}$.) Suppose $R_\mu T = 0$; then $DR_\mu^* \varphi|_{X=0} = 0$ for every $\varphi \in C^\infty(\Xi_0)$. Lemma 7.3 shows this is impossible. This proves Theorem 7.1. □

Before proving Lemma 7.3 we must review some notation and facts concerning polynomials and differential operators on $\mathfrak{p}$. Let $S(\mathfrak{p}^*)$ denote the polynomial algebra of $\mathfrak{p}$. Let $Q(X) = \langle X, X \rangle$. Let J be the subalgebra generated by Q. J is the algebra of K-invariant polynomials. As before, let $\mathscr{H}$ be the space of harmonic polynomials. An important result of Rallis (see [9]) says $S(\mathfrak{p}^*) = \mathscr{H} \otimes J$. As in §4, $\mathscr{H} = \bigoplus \mathscr{H}_\delta$ and we consider elements in $\mathscr{H}$ as either polynomials on $\mathfrak{p}$ or functions on K/M. Let $S(\mathfrak{p})$ be the symmetric algebra of $\mathfrak{p}$. We will also think of $S(\mathfrak{p})$ as the algebra of constant coefficient differential operators on $\mathfrak{p}$. Of course, K acts on $S(\mathfrak{p})$ and the Killing form induces a K-equivariant isomorphism, Ψ, between $S(\mathfrak{p}^*)$ and $S(\mathfrak{p})$. For each $\delta \in \widehat{K}_M$ let V_δ be the representation space with unitary structure $\langle\ ,\ \rangle_\delta$. Fix a K-equivariant isomorphism $j_\delta \colon V_\delta \to \mathscr{H}_\delta$. We will drop the subscript δ in places where there is no danger of confusion. For each δ fix a unit vector $v_\delta \in V_\delta$ which is M invariant and such that $j(v_\delta) = \varphi_\delta$. Thus, $j_\delta(v)(k) = \langle v, k \cdot v_\delta \rangle$. Finally, let $D_\delta = \Psi(\varphi_\delta)$ and $L = \Psi(Q) =$ the Laplacian.

To summarize,

$$\begin{aligned}
&Q(X) = \langle X, X\rangle, \quad J = \mathbb{C}[Q], \qquad L = \Psi(Q), \\
&\Psi\colon S(\mathfrak{p}^*) \to S(\mathfrak{p}); \quad S(\mathfrak{p}^*) = \mathscr{H} \otimes J, \quad \mathscr{H} = \bigoplus \mathscr{H}_\delta, \\
&j_\delta\colon V_\delta \to \mathscr{H}_\delta, \quad v_\delta = M \text{ invariant unit vector}, \\
&j_\delta(v_\delta) = \varphi_\delta, \quad \Psi(\varphi_\delta) = D_\delta.
\end{aligned}$$

Lemma 7.2. (1) *Suppose* $P \in \mathscr{H}_\delta$ *and* $D = \Psi(P)$. *Then (bar indicates complex conjugation)*

$$\int_K \overline{P(K)}(k^{-1}\cdot D)\,dk = cD_\delta$$

for some $c \neq 0$.

(2) *Suppose* $P \in \mathscr{H}_\delta$, $P' \in \mathscr{H}_{\delta'}$, $D = \Psi(P)$, *and* $\delta \neq \delta'$. *Then*

$$\int_K \overline{P'(k)}(k^{-1}\cdot D)\,dk = 0.$$

Proof. (1) Choose $v \in V_\delta$ such that $j(v) = P$. (So, $P(k) = \langle v, k\cdot v_\delta\rangle$.) Thus,

$$\begin{aligned}
\int_K \overline{P(k)}(k^{-1}\cdot D)\,dk &= \Psi\circ j\left(\int_K \overline{\langle v, k\cdot v_\delta\rangle}k^{-1}\cdot v\,dk\right) \\
&= \Psi\circ j\left(\int_K \overline{\langle k\cdot v, v_\delta\rangle}k\cdot v\,dk\right) \\
&= \Psi\circ j(c_1\langle v, v\rangle v_\delta) \qquad (c_1 = (\dim\delta)^{-1}) \\
&= c_1\langle v, v\rangle D_\delta.
\end{aligned}$$

The third equality is just the Schur orthogonality relations.

Part (2) is also just the orthogonality relations. □

Recall, we can consider $S(\mathfrak{p})$ to be the algebra of constant coefficient differential operators.

Lemma 7.3. *Suppose* $D = \sum_{\delta,j} D_{\delta,j}L^j$, *where each* $D_{\delta,j} \in \mathscr{H}_\delta$. *Then there is a* $\varphi \in C^\infty(\Xi_0)$ *such that* $DR_\mu^*\varphi(0) \neq 0$.

Proof. Choose a δ such that $D_{\delta,j} \neq 0$ for some j. Let N be the largest j for which this is true. Take $P \in \mathscr{H}_\delta$ such that $\Psi(P) = D_{\delta,N}$. Define φ by

$$\varphi(k\cdot(rH_0 + \mathfrak{q})) = \overline{P(k)}g(rH_0) = \overline{P(k)}r^{d_\delta + 2N}.$$

We get

$$\begin{aligned}
DR_\mu^*\varphi(0) &= D\left(\int_K \varphi(X + k\cdot\mathfrak{q})\mu(T_Xk)\,dk\right)_{X=0} \\
&= D\left(\int_K \varphi(k\cdot(P_\mathfrak{a}(k^{-1}\cdot X) + \mathfrak{q})\mu(kT_{k^{-1}\cdot X})\,dk\right)_{X=0} \\
&= \int_K \overline{P(k)}(k^{-1}\cdot D)(g(P_\mathfrak{a}(X)\mu(T_X))_{X=0}\,dk \\
&= (c_ND_\delta L^N + c_{N-1}D_\delta L^{N-1} + \cdots + c_0D_\delta)(g(P_\mathfrak{a}(X))\mu(T_X))_{X=0}.
\end{aligned}$$

The third equality follows from the K invariance of μ and the definition of the action of K on differential operators. The last equality follows from Lemma 7.2, which guarantees $c_N \neq 0$.

The polynomial (in X) $g \circ P_{\mathfrak{a}}(X)$ is homogeneous of degree $d_\delta + 2N$. Since D_δ is homogeneous of degree d_δ and L is homogeneous of degree 2 the last equation above implies $DR^*_\mu \varphi(0) = c_N \mu(T_0)(D_\delta L^N g \circ P_{\mathfrak{a}})(0)$.

We know $\varphi_\delta(H_0) = 1$ and φ_δ is homogeneous of degree d_δ. Thus, for $Y \in \mathfrak{q}$ we have $\varphi_\delta(rH_0 + Y) = r^d + R(rH_0 + Y)$ where $R(Y) = 0$. This shows (as an element of $S(\mathfrak{p})$) $D_\delta \in H_0^d + S(\mathfrak{p})\mathfrak{q}$. Similarly, $L \in H_0^2 + S(\mathfrak{p})\mathfrak{q}$. Thus, $D_\delta L^N g \circ P_{\mathfrak{a}}(rH_0 + T) = (d_\delta + 2N)!$. This proves the lemma.

Corollary 7.4. *If* μ *satisfies the hypothesis of Corollary* 6.5 *then*

(1) *If* $T \in \mathscr{E}'(\Xi_{\mathbb{F}})$ *and* $\operatorname{supp} R_\mu T \subset B^R$ *then* $\operatorname{supp} T \subset B^R$.

(2) ${}_{\mathbb{F}}R_\mu : \mathscr{E}'(X_0) \to \mathscr{E}'(\Xi_{\mathbb{F}})$ *is injective.*

Proof. Part (1) follows from Lemma 6.4 and the proof of Theorem 7.1.

As in the proof of Theorem 7.1, part (2) reduces to showing ${}_{\mathbb{F}}R_\mu$ is injective on distributions supported at the origin. This follows from the proof of Theorem 7.1 using Lemma 6.4. □

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